Fourier Series Homework
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Problem 1

Let $f : X \to Y$ be a map of metric spaces.
(a) Write the definition of f being uniformly continuous.
(b) Write the definition of X being compact.
(c) Suppose that X is compact, and that f is continuous. Show that f is uniformly continuous.

Problem 2

Define the sequence of functions $\{f_n\}$ on the interval $[0, 1]$ by $f_n(x) = x^n$.
(a) What function does f_n converge to? *Hint: It's piecewise.*
(b) Show that this convergence is uniform on the interval $[0, 1 - \alpha]$ for $0 < \alpha < 1$.

Problem 3

Let $f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \pi \\ 1 & \text{if } -\pi < x < 0 \end{cases}$, where $f(x + 2\pi) = f(x)$, i.e., f has period 2π. Calculate the Fourier series of f.

Problem 4

Recall the definition of $D_n(x)$, the Dirichlet kernel of index n. Show that the n-th partial Fourier sum has the following characterization:

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) D_n(x - u) du,$$

and use this to show that $S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x - t) D_n(t) dt$.