Math Academy Guest Lecture: Fourier Series and the Basel Problem

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The purpose of the following two lectures is to develop sufficiently the theory of Fourier series so that we may solve the Basel problem:

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

and compute similar infinite sums.

Fourier series are instances of what are called *trigonometric polynomials:*

Definition: Trigonometric polynomials

A *trigonometric polynomial* is a finite sum

$$f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx))$$

for complex constants $a_0, a_1, ..., a_N$ and $b_1, ..., b_n$.

Using Euler's formula, we may also write such a polynomial in the form

$$f(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

where the coefficients are derived as

$$c_n = \begin{cases} \frac{a_n - ib_n}{2} & \text{if } n > 0 \\ \frac{a_n + ib_n}{2} & \text{if } n < 0 \\ 0 & \text{if } n = 0. \end{cases}$$

Notice that any trigonometric polynomial must have period 2π. For two Riemann-integrable functions f, g on an interval $[a, b]$, we know that their inner product is defined as

$$\int_a^b f(x) g(x) dx.$$

Now since we are dealing with periodic functions, our interval is now $[-\pi, \pi]$.

Let's compute some inner products of cosines and sines to develop some intuition for Fourier series. Firstly,

$$\langle \cos(nx), \cos(m, x) \rangle = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \neq 0 \\ 2\pi & \text{if } n = 0 = m. \end{cases}$$

(Write out the details) Similarly,

$$\langle \sin(nx), \sin(mx) \rangle = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \neq 0 \\ 2\pi & \text{if } n = 0 = m. \end{cases}$$

We also have that $\langle 1, \cos(nx) \rangle = 0$ and $\langle 1, \sin(nx) \rangle = 0$ for all n.

These facts tell us that if we have a trigonometric polynomial

$$f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)),$$

we find that $\langle f(x), \cos(nx) \rangle = \pi a_n$ and $\langle f(x), \sin(nx) \rangle = \pi b_n$. This motivates the following definition:

Definition: Fourier Series of a Function

Let f be a real (or complex) valued function with period 2π. Then the *Fourier series of* f is the infinite series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

for the constants $a_n = \frac{1}{\pi} \langle f(x), \cos(nx) \rangle$ and $b_n = \frac{1}{\pi} \langle f(x), \sin(nx) \rangle$. In this case we write

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)),$$

but beware, this does not mean that $f(x)$ is equal to its Fourier series!

Let's compute some examples now!

Exercise

Calculate the Fourier series of $f(x) = x$ on the interval $[-\pi, \pi]$.

Exercise

Calculate the Fourier series of

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \pi \\ 1 & \text{if} -\pi < x < 0 \end{cases}$$

on the interval $[-\pi, \pi]$.

Fourier series are not restricted to functions with period 2π. Suppose we have a (Riemann integrable) function of period $2L$. Then the arguments of our trigonometric functions become $\frac{n\pi x}{L}$ and we have

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L} \right) + b_n \sin\left(\frac{n\pi x}{L} \right) \right)$$

with $a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(\frac{n\pi x}{L}) dx$ and $b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(\frac{n\pi x}{L}) dx$.

Note that even functions will have no sine terms, and odd functions will have no cosine terms. (Why?) We don't yet know about the convergence of Fourier series, but they are in a sense the best approximation of a function using sines and cosines:

Theorem

Let $\{\varphi_1, \varphi_2, ..., \varphi_n\}$ be an orthonormal set of functions and $\{c_1, .., c_n\}$ the Fourier coefficients of a piecewise continuous function f with respect to this set ($c_i = \int f(x)\varphi_i(x)dx$). Taking $S_n(x) = \sum_{i=1}^{n} c_i\varphi_i(x)$ and $T_n(x) = \sum_{i=1}^{n} d_i\varphi_i(x)$ for *any* constants $\{d_i\}$, we have

$$\langle f - S_n, f - S_n\rangle \leq \langle f - T_n, f - T_n\rangle.$$

Equality holds if and only if $c_i = d_i$.

Proof: We have that

$$\begin{aligned}\langle f - T_n, f - T_n\rangle &= \int_{-\pi}^{\pi} [f(x) - T_n(x)][f(x) - T_n(x)]dx \\ &= \int_{-\pi}^{\pi} [f(x)]^2 - 2f(x)T_n(x) + [T_n(x)]^2 dx.\end{aligned}$$

Since $\{\varphi_i\}$ was an orthonormal set, we have that

$$\int_{-\pi}^{\pi} [T_n(x)]^2 dx = \int_{-\pi}^{\pi} \left[\sum_{i=1}^{n} d_i\varphi_i(x)\right]\left[\sum_{i=1}^{n} d_i\varphi_i(x)\right] = \sum_{i=1}^{n} d_i^2.$$

By the definition of the Fourier coefficients, we know that

$$2\int_{-\pi}^{\pi} f_n(x)T_n(x) = 2\sum_{i=1}^{n} c_i d_i.$$

Thus the inner product $\langle f - T_n, f - T_n\rangle$ is

$$\int_{-\pi}^{\pi} [f(x)]^2 dx - 2\sum_{i=1}^{n} d_i c_i + \sum_{i=1}^{n} d_i^2,$$

which looks like a square. We manipulate to find that

$$\langle f - T_n, f - T_n\rangle = \int_{-\pi}^{\pi} [f(x)]^2 dx - \sum_{i=1}^{n} c_i^2 + \sum_{i=1}^{n} (c_i - d_i)^2.$$

Then by computing the inner product $\langle f - S_n, f - S_n\rangle$, we find that it equals $\int_{-\pi}^{\pi} [f(x)]^2 dx - \sum_{i=1}^{n} c_i^2$. Since the last term in the inner product involving T_n is non-negative, we have what we wanted.

□

Corollary: Bessel's Inequality

While computing inner products in the last proof, we found all we need to prove that

$$\sum_{i=1}^{\infty} c_i^2 \leq \int_{-\pi}^{\pi} [f(x)]^2 dx.$$

Proof: We have that

$$\int_{-\pi}^{\pi} [f(x)]^2 dx - \sum_{i=1}^{n} c_i^2 = \langle f - S_n, f - S_n \rangle \geq 0 \qquad \forall n.$$

Therefore

$$\lim_{n\to\infty} \sum_{i=1}^{n} c_i^2 = \sum_{i=1}^{\infty} c_i^2 \leq \int_{-\pi}^{\pi} [f(x)]^2 dx.$$

□

This result will be very important when we prove Parseval's equality which allows us to calculate infinite sums exactly.

Initially our functions $\cos(nx)$ and $\sin(nx)$ are not orthonormal. But they are only off by a constant multiple, i.e.,

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{\sin(x)}{\sqrt{\pi}}, \dots \right\}$$

is an orthonormal set. Then we can relate the coefficients c_i coming from this orthonormal set to the a_i and b_i as

$$\sum_{i=1}^{\infty} c_i^2 = \pi \left(a_0^2 + \sum_{i=1}^{\infty} (a_i^2 + b_i^2) \right).$$

Definition: n-th partial Fourier sum

Suppose that f is a piecewise continuous function of period 2π and

$$f \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)),$$

then the *n-th partial Fourier sum* is $S_n(x) = a_0 + \sum_{i=1}^{n} (a_n \cos(nx) + b_n \sin(nx))$.

In order to study these partial sums, we make the following definition.

Definition: Dirichlet kernel of index n

The *Dirichlet kernel* of index n is the function $D_n(x)$ defined as

$$D_n(x) = \frac{1}{2} + \cos(x) + \cos(2x) + \dots + \cos(nx).$$

Theorem: Characterization of partial Fourier sums

Let f be a piecewise continuous function of period 2π. Then

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)D_n(x-u)du$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)dt.$$

Proof: Homework. :)

□

Definition: n-th Cesaro sum

The n-th Cesaro sum of the function f is defined as

$$\sigma_n(x) = \frac{S_0(x) + S_1(x) + \dots + S_n(x)}{n+1},$$

where the S_i are the partial Fourier sums.

Definition: Fejer kernel

The *Fejer kernel of index* n is the function

$$K_n(x) = \frac{D_0(x) + D_1(x) + \dots + D_n(x)}{n+1}.$$

It has the following properties (which you can prove yourself)

1. $\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1$
2. $K_n(x) \leq \frac{1}{(n+1)(1-\cos(\delta))}$ for $0 < \delta < |x| < \pi$.

Theorem: Uniform convergence of Cesaro sums

If f is continuous with period 2π, then σ_n converges uniformly to f.

Proof: We have that

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)dt,$$

implying that

$$\sigma_n(t) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)K_n(t)dt.$$

Then since $\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(t)dt = 1$, we have that $f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)K_n(t)dt$ (note the variables).

Now we have that

$$\sigma_n(x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} [f(x-t) - f(x)]K_n(t)dt.$$

For $0 < \delta < \pi$, this is equal to

$$\begin{aligned}&\frac{1}{\pi}\int_{-\pi}^{-\delta}[f(x-t)-f(x)]K_n(t)dt\\&+\frac{1}{\pi}\int_{-\delta}^{\delta}[f(x-t)-f(x)]K_n(t)dt\\&+\frac{1}{\pi}\int_{\delta}^{\pi}[f(x-t)-f(x)]K_n(t)dt.\end{aligned}$$

We work to bound each of these integrals. Firstly,

$$\frac{1}{\pi}\int_{\delta}^{\pi}|f(x-t)-f(x)\|K_n(t)|dt\leq\frac{1}{\pi}\cdot\frac{1}{n+1}\cdot\frac{1}{1-\cos(\delta)}\int_{\delta}^{\pi}|f(x-t)-f(x)|\ dt.$$

Then the continuity of f implies that it is bounded on $[-\pi,\pi]$. We can then say that $|f(x-t)-f(x)|\leq 2M$ for some M. So

$$\frac{1}{\pi}\int_{\delta}^{\pi}|f(x-t)-f(x)\|K_n(t)|dt\leq\frac{2M}{(n+1)(1-\cos(\delta))}.$$

Similarly

$$\frac{1}{\pi}\int_{-\pi}^{-\delta}|f(x-t)-f(x)\|K_n(t)|dt\leq\frac{2M}{(n+1)(1-\cos(\delta))}.$$

Now we bound the middle integral. Since f is continuous on a compact set, it is uniformly continuous (HW). So given $\varepsilon>0$, there is some $\delta>0$ such that

$$|f(x)-f(y)|<\frac{\varepsilon}{2}$$

for all $x,y\in[-\pi,\pi]$ with $|x-y|<\delta$. Therefore $|t|<\delta$ implies $|f(x-t)-f(x)|<\frac{\varepsilon}{2}$. So

$$\int_{-\delta}^{\delta}|f(x-t)-f(x)\|K_n(t)|dt\leq\frac{\varepsilon}{2}\int_{-\delta}^{\delta}|K_n(t)|\ dt\leq\frac{\varepsilon}{2}\int_{-\pi}^{\pi}K_n(t)dt=\frac{\varepsilon}{2}.$$

Then since we may take the other two integrals as small as we want, we have uniform convergence.

□

Theorem: Parseval's Theorem (the main result)

Let f be a continuous function of period 2π. If $f\sim\sum_{n=0}^{\infty}c_ne^{inx}$ then $\sum_{n=0}^{\infty}c_n^2=\int_{-\pi}^{\pi}[f(x)]^2dx$. Then by our earlier relationship between the c's and a's and b's,

$$a_0^2+\sum_{n=1}^{2}(a_n^2+b_n^2)=\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2dx.$$

Proof: We have that $\langle f-S_n,f-S_n\rangle\leq\langle f-\sigma_n,f-\sigma_n\rangle$ for all n. But $\langle f-\sigma_n,f-\sigma_n\rangle=\int_{-\pi}^{\pi}[f(x)-\sigma_n(x)]^2dx$. Since $\sigma_n\to f$ uniformly,

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}[f(x)-\sigma_n(x)]^2dx=\int_{-\pi}^{\pi}\lim_{n\to\infty}[f(x)-\sigma_n(x)]^2dx=0.$$

Then $\lim_{n\to\infty} \int_{-\pi}^{\pi} [f(x) - S_n(x)]^2 dx = 0$ by the squeeze theorem. And we calculated earlier that $\int_{-\pi}^{\pi} [f(x) - S_n(x)]^2 dx = \int_{-\pi}^{\pi} [f(x)]^2 dx - \sum_{i=0}^{n} c_i^2$. So the limit is

$$\int_{-\pi}^{\pi} [f(x)]^2 dx - \sum_{i=0}^{\infty} c_i^2 = 0,$$

which proves the theorem.

□

Now by calculating the Fourier series of $f(x) = x$ on $[-\pi, \pi]$, we may use Parseval's theorem to solve the Basel problem.